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Nonlinear periodic convection in double-diffusive systems

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We study two examples of two-dimensional nonlinear double-diffusive convection (thermohaline convection, and convection in an imposed vertical magnetic field) in the limit where the onset of marginal overstability just precedes the exchange of stabilities. In this limit nonlinear solutions can be found analytically. The branch of oscillatory solutions always terminates on the steady solution branch. If the steady solution branch is subcritical this occurs when the period of the oscillation becomes infinite, while if it is supercritical, it occurs via a Hopf bifurcation. A detailed discussion of the stability of the oscillations is given. The results are in broad agreement with the larger-amplitude results obtained previously by numerical techniques.

1. Introduction

Although nonlinear convection has been much studied in recent years, there are few exact solutions available. While this perhaps reflects the difficulty of the subject, it does not reduce the need for and the usefulness of such solutions. Guided by this consideration, we examine in this paper two frequently discussed problems, thermohaline convection and magnetoconvection, with a view to selecting a parameter regime that would make analytical solutions accessible.

Two-dimensional nonlinear thermohaline convection in the Boussinesq approximation has been considered by Veronis (1968*b*) and Huppert & Moore (1976), who obtained a variety of solutions by a numerical integration of the governing equations. More recently, Da Costa, Knobloch & Weiss (1981) utilized a five-mode truncation originally suggested by Veronis (1965) to obtain nonlinear solutions that were in good qualitative agreement with the numerical results of Huppert & Moore. Moreover, the truncated equations could be solved in part analytically, and the topology of the solutions as well as their stability could be investigated. Two-dimensional Boussinesq convection in an imposed vertical magnetic field is a closely related problem although it allows for a more bewildering variety of nonlinear effects (Weiss 1981*a*,*b*). Knobloch, Weiss & Da Costa (1981) again found that a five mode truncation provides nonlinear solutions in good qualitative agreement with the numerical solutions and with the same wealth of phenomena. Although the work on the truncated modal equations provided a number of useful results, such as a direct determination of the stability properties of the branch of steady solutions as a function of the applied Rayleigh number, no rational approximations to nonlinear solutions of the exact equations were obtained. Such information

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has been thus far confined to the results of modified perturbation theory (Veronis 1968*b*; Huppert & Moore 1976; Knobloch *et al.* 1981).

Both thermohaline convection and magneto-convection are examples of double-diffusive convection. As such they admit both steady convection solutions and oscillatory solutions. In the parameter regime when the Rayleigh number for marginal overstability $R_T^{(o)}$ is just smaller than that for the exchange of stabilities, $R_T^{(e)}$, oscillatory convection sets in first, but in view of the proximity of $R_T^{(o)}$ to $R_T^{(e)}$ its amplitude remains small. Hence although the oscillations are fully nonlinear, they are accessible to a perturbation analysis based on powers of $[R_T^{(e)} - R_T^{(o)}]^{\frac{1}{2}}$.

In this paper we carry out such an analysis, and find that in this parameter regime asymptotically correct solutions can be found in closed form. We find it convenient to make use of the truncated modal equations of Da Costa *et al.* (1981) and Knobloch *et al.* (1981) since they can be shown to be exact consequences of the full equations at leading order in the expansion parameter. The procedure yields an evolution equation whose solutions are Jacobian elliptic functions, and an integrability condition whose solution determines the Rayleigh number corresponding to each solution. In this way a nonlinear amplitude–Rayleigh-number diagram can be constructed. A rather different procedure was used by Rubenfeld & Siegman (1977) in their study of the loop model of thermohaline convection.

We find that the solutions are qualitatively similar to those obtaining at larger amplitudes. Thus when the steady solution branch is subcritical, the branch of oscillatory solutions terminates on it with the period of the oscillations becoming infinite (Da Costa *et al.* 1981), while when it is supercritical the oscillatory branch terminates on it via a Hopf bifurcation (Knobloch *et al.* 1981). The stability properties of the oscillatory solutions near this bifurcation are not yet fully understood. Knobloch *et al.* (1981) found, after a careful study, that the small-amplitude oscillations near the bifurcation point are unstable with a rather slow growth rate. On the other hand Weiss (1981*a*,*b*), studying the full equation, found regions in the neighbourhood of the bifurcation for which the oscillations appear to be stable, though stability is lost at larger amplitudes. In the model presented here, the oscillatory mode is neutrally stable at the bifurcation point, and unstable everywhere else at leading order in the expansion scheme. Thus if the oscillations have a region of stability, its size must be determined at higher order in the expansion, and this is too lengthy to attempt here. The more exotic phenomena (period doubling bifurcations, aperiodic oscillations) found at larger amplitudes are not present in this regime.

In §2 we treat the simpler thermohaline problem in some detail, and employ the same techniques in §3 to study the magnetoconvection problem. Comparison is made in each case with the larger amplitude numerical results. Brief conclusions are presented in §4.

2. Thermohaline convection

2.1. *Basic equations*

We consider two-dimensional thermohaline convection in a horizontal layer of fluid confined between the planes $z = 0$ and $z = h$, and adopt the Boussinesq approximation. The density is taken to be $\rho = \rho_0(1-\alpha(T-T_0)+\beta(S-S_0))$ where T is the temperature and S the solute concentration, and $\alpha, \beta > 0$. We restrict our attention to the case in which the solute gradient is stabilizing and the temperature gradient destabilizing. Thus

$$T = T_0 + \Delta T(1 - z + \Theta(x,z)), \quad S = S_0 + \Delta S(1 - z + \Sigma(x,z)), \tag{2.1}$$

where $\Delta T, \Delta S > 0$. We then scale the velocity $u(x,z,t) \equiv (-\partial_z\psi, 0, \partial_x\psi)$ where ψ is a stream function, with respect to the thermal diffusion velocity κ_T/h, where κ_T is the thermal conductivity. We also scale lengths with h and time scales with h^2/κ_T and arrive at the following dimensionless equations for Θ, Σ and ψ:

$$\sigma^{-1}[\partial_t \nabla^2\psi + J(\psi, \nabla^2\psi)] = R_T\,\partial_x\Theta - R_S\,\partial_x\Sigma + \nabla^4\psi, \tag{2.2}$$

$$\partial_t\Theta + J(\psi, \Theta) = \partial_x\psi + \nabla^2\Theta, \tag{2.3}$$

$$\partial_t\Sigma + J(\psi, \Sigma) = \partial_x\psi + \tau\nabla^2\Sigma, \tag{2.4}$$

where

$$\sigma = \nu/\kappa_T, \tau = \kappa_S/\kappa_T, \quad R_T = g\alpha\Delta T h^3/\kappa_T\nu, \quad R_S = g\beta\Delta S h^3/\kappa_T\nu, \tag{2.5}$$

and ν, κ_S, g are respectively the kinematic viscosity, solute diffusivity and acceleration due to gravity. These equations are in the standard form used by Huppert & Moore (1976) and Da Costa *et al.* (1981). As in these studies, we use the simplest boundary conditions: at $z = 0, 1$ there is no normal velocity or tangential stress, and no horizontal gradients of temperature or solute concentration, while in the horizontal direction, we impose periodic boundary conditions appropriate to a cell of (dimensionless) width λ. Thus

$$\psi = \partial^2_{zz}\psi = \Theta = \Sigma = 0, \quad z = 0, 1 \tag{2.6}$$

and

$$\psi = \partial^2_{xx}\psi = \partial_x\Theta = \partial_x\Sigma = 0, \quad x = 0, \lambda. \tag{2.7}$$

The system described above has the static (conductive) solution $\psi = \Theta = \Sigma = 0$. However, if R_T is sufficiently large, this state is unstable. Linear stability theory (Baines & Gill 1969; Huppert & Moore 1976) shows that, as R_T is increased, the first instability to occur is either

$$\left.\begin{array}{ll} & (a)\ \text{a direct mode at } R_T = R_T^{(e)} \text{ if } \tau \geqslant 1 \text{ or } R_S \leqslant R_{SC}, \\ \text{or} & (b)\ \text{an oscillatory (overstable) mode at } R_T = R_T^{(o)} \text{ if } \tau < 1 \text{ and } R_S > R_{SC}, \end{array}\right\} \tag{2.8}$$

where $R_T^{(e)}$, $R_T^{(o)}$, R_{SC} are defined in terms of the new parameters $r_T^{(e)}, r_T^{(o)}, r_{SC}$ by

$$\left.\begin{aligned} (R_T, R_S) &= \frac{\lambda^2 p^3}{\pi^2}(r_T, r_S), \\ p &= \pi^2(1+\lambda^{-2}), \end{aligned}\right\} \tag{2.9}$$

with and

$$\left.\begin{aligned} r_T^{(e)} &= 1 + r_S/\tau, \\ r_T^{(o)} &= 1 + \frac{\Delta\tau}{\sigma} + \left(\frac{\sigma+\tau}{\sigma+1}\right) r_S, \\ r_{SC} &= \frac{\tau^2(\sigma+1)}{\sigma(1-\tau)}, \\ \Delta &= 1 + \sigma + \tau. \end{aligned}\right\} \tag{2.10}$$

where $\Delta = 1 + \sigma + \tau$.

In the following we shall assume that the parameters are such that overstable modes are possible, in which case $r_T^{(o)}$ is always less than $r_T^{(e)}$.

Because of the nonlinearity of the equations, finite-amplitude periodic oscillations can exist when r_T is in a (not necessarily small) neighbourhood of $r_T^{(o)}$, while a family of steady solutions exists in a neighbourhood of $r_T^{(e)}$. The relationship between the amplitude of convection and r_T is known in each case from modified perturbation theory (Huppert & Moore 1976). For example, for the steady solutions, the amplitude a is related to r_T by

$$r_T = r_T^{(e)} + r_2^{(e)}\, a^2 + O(a^4), \tag{2.11}$$

where

$$\sigma\tau^3 r_2^{(e)} = \sigma\tau^3 + \sigma(\tau^2-1)\, r_S = -\tau^2\Delta - (1+\sigma)(1+\tau)\,\omega_0^2, \tag{2.12}$$

and ω_0 is the frequency of infinitesimal oscillations at $r_T = r_T^{(o)}$. Thus $r_2^{(e)} < 0$ if oscillations are possible, and the steady solutions are subcritical. A similar expansion holds in the neighbourhood of $r_T^{(o)}$ for the oscillatory branch. For most values of the parameters these expansions rapidly become inaccurate and the equations (2.2)–(2.4) have to be integrated numerically. However in the case $0 < r_S - r_{SC} \ll 1$ the behaviour of both branches of solutions can be elucidated analytically, even when the structure of the oscillatory branch deviates markedly from that of the linearized theory. This is because in the neighbourhood of $r_S = r_{SC}$ the oscillations have a low frequency: at $r_S = r_{SC}$ the linear stability problem for the static solution has a double zero eigenvalue. The orbital stability of the solutions can also be determined as a function of their amplitude. In particular, we shall describe the manner in which the finite-amplitude oscillatory solutions form a single parameter family which comprises both the infinitesimal oscillation at $r_T = r_T^{(o)}$ and a heteroclinic orbit of indefinitely long period that joins the (unstable) steady solution branch that bifurcates from $r_T^{(e)}$. Such a family has been found by Da Costa *et al.* using numerical methods on a truncated set of modal equations.

2.2 *The problem in the limit* $r_S \to r_{SC}$

We now suppose that r_S is close to r_{SC} and that r_T is close to $r_T^{(o)}$ and $r_T^{(e)}$. Specifically, we write

$$r_S = r_{SC} + \epsilon^2, \quad \epsilon^2 \ll 1 \tag{2.13}$$

and from (2.10) then find that

$$\left.\begin{aligned} r_T^{(o)} &= \frac{\sigma+\tau}{\sigma(1-\tau)} + \left(\frac{\sigma+\tau}{\sigma+1}\right)\epsilon^2, \\ r_T^{(e)} &= \frac{\sigma+\tau}{\sigma(1-\tau)} + \frac{1}{\tau}\,\epsilon^2. \end{aligned}\right\} \tag{2.14}$$

We therefore set

$$r_T = \frac{\sigma+\tau}{\sigma(1-\tau)} + \mu\epsilon^2, \tag{2.15}$$

where $\mu = O(1)$. Since r_T is close to $r_T^{(o)}$, the amplitude of the motion is small (and previous studies show that it is $O(\epsilon)$) as are the amplitudes of Θ and Σ. From linearized theory we also find that the oscillation frequency ω_0 is given by

$$\omega_0 = \epsilon\left[\frac{\sigma(1-\tau)}{1+\sigma}\right]^{\frac{1}{2}} \tag{2.16}$$

so that it is appropriate to define a new time scale

$$t^* = \epsilon p t, \tag{2.17}$$

where the order unity factor p (cf. (2.9)) is introduced for convenience. To avoid any non-uniformity in the perturbation scheme, we will require that τ and $(1-\tau)$ are both of order unity. Then the stream function ψ may be expanded in powers of ϵ; the structure of the nonlinear terms and the results of modified perturbation theory imply that the correct expansion is of the form

$$\psi = \epsilon\psi_1 + \epsilon^3\psi_3 + \dots, \tag{2.18}$$

where

$$\psi_1 \propto \sin(\pi\chi/\lambda)\sin\pi z, \quad \psi_3 \propto \sin(\pi\chi/\lambda)\sin 3\pi z, \quad \text{etc.};$$

Θ and Σ can be expanded similarly. We may write

$$\psi = \frac{2}{\pi}(2p)^{\frac{1}{2}}\lambda\{\epsilon\sin(\pi\chi/\lambda)\sin\pi z\, a_1(t^*) + \epsilon^3\sin(\pi\chi/\lambda)\sin 3\pi z\, a_3(t^*) + \dots\}, \tag{2.19}$$

$$\Theta = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\{\epsilon\cos(\pi\chi/\lambda)\sin\pi z\, b_1(t^*) + \epsilon^3\cos(\pi\chi/\lambda)\sin 3\pi z\, b_3(t^*) + \dots\}$$
$$-\frac{1}{\pi}\epsilon^2\sin 2\pi z\, c(t^*) + \dots, \tag{2.20}$$

$$\Sigma = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\{\epsilon\cos(\pi\chi/\lambda)\sin\pi z\, d_1(t^*) + \epsilon^3\cos(\pi\chi/\lambda)\sin 3\pi z\, d_3(t^*) + \dots\}$$
$$-\frac{1}{\pi}\epsilon^2\sin 2\pi z\, e(t^*) + \dots, \tag{2.21}$$

where the $O(\epsilon^2)$ terms in Θ and Σ reflect the tendency of the heat and solute to be confined to boundary layers near $z = 0, 1$.

The expansion can be continued indefinitely, but the terms written are more than sufficient for our purpose. If the expansions (2.19)–(2.21) are substituted into the governing equations (2.2)–(2.5) and r_T and r_S are written in terms of ϵ using (2.13), (2.15), we obtain a hierarchy of ordinary differential equations for the modal amplitudes, of which the first five are (Veronis 1965, 1968*b*; Da Costa *et al.* 1981)

$$\epsilon a_1' = \sigma\left[-a_1 + b_1\left\{\frac{\sigma+\tau}{\sigma(1-\tau)} + \mu\epsilon^2\right\} - d_1\left\{\frac{\tau^2(\sigma+1)}{\sigma(1-\tau)} + \epsilon^2\right\}\right] + O(\epsilon^4), \tag{2.22}$$

$$\epsilon b_1' = -b_1 + a_1 - \epsilon^2 a_1 c + O(\epsilon^4), \tag{2.23}$$

$$\epsilon c' = \varpi[-c + a_1 b_1] + O(\epsilon^2), \tag{2.24}$$

$$\epsilon d_1' = -\tau d_1 + a_1 - \epsilon^2 a_1 e + O(\epsilon^4), \tag{2.25}$$

$$\epsilon e' = \varpi(-\tau e + a_1 d_1) + O(\epsilon^2), \tag{2.26}$$

where the prime denotes differentiation with respect to t^* and $\varpi = 4\pi^2/p\ (O < \varpi < 4)$. The $O(\epsilon^2)$, $O(\epsilon^4)$ corrections, involving a_3, b_3, and d_3 are not required in the present calculation.

We now seek a solution to this system as an expansion in powers of ϵ. Before doing so we note that Da Costa *et al.* solved equations (2.22)–(2.26) ignoring the higher-order terms, even when $\epsilon = O(1)$; it must be emphasized that the analysis to follow includes no such arbitrary truncations.

2.3. *Solution in powers of* ϵ

For brevity we now drop the subscripts on the leading order amplitudes. Equations (2.22)–(2.26) show that for $\epsilon \ll 1$,

$$a \simeq b \simeq \tau d, \quad c \simeq ab, \quad e \simeq \tau^{-1}ad. \tag{2.27}$$

Hence if we set

$$b = a + \epsilon g, \quad d = a/\tau + \epsilon h, \quad c = a^2 + \epsilon k, \quad e = a^2/\tau^2 + \epsilon l, \tag{2.28}$$

it then follows from (2.23) that

$$g = -b' - \epsilon ac + O(\epsilon^3). \tag{2.29}$$

Elimination of b using (2.28a) and repeated use of this equation and of (2.29) gives

$$\left.\begin{aligned} g &= a' - \epsilon g' - \epsilon ac + O(\epsilon^3), \\ &= -a' + \epsilon(b'' - ac) + \epsilon^2(ac)' + O(\epsilon^3), \\ &= -a' + \epsilon(a'' - ac) + \epsilon^2 g'' + \epsilon^2(ac)' + O(\epsilon^3), \\ &= -a' + \epsilon(a'' - a^3) - \epsilon^2(ak + a''' - (a^3)') + O(\epsilon^3). \end{aligned}\right\} \tag{2.30}$$

The same procedure applied to k gives

$$k = ag - \frac{2aa'}{\varpi} + O(\epsilon) = -aa'\left(1 + \frac{2}{\varpi}\right) + O(\epsilon) \tag{2.31}$$

so that, finally, using (2.28c) to substitute for k we obtain

$$g = -a' + \epsilon(a'' - a^3) - \epsilon^2\left(a''' - \left(4 + \frac{2}{\varpi}\right)a^2 a'\right) + O(\epsilon^3). \tag{2.32}$$

Similarly,

$$h = -\frac{a'}{\tau^2} + \frac{\epsilon}{\tau^3}(a'' - a^3) - \frac{\epsilon^2}{\tau^4}\left(a''' - \left(4 + \frac{2}{\varpi}\right)a^2 a'\right) + O(\epsilon^3). \tag{2.33}$$

The expressions obtained for g and h can now be substituted into (2.28a, b) and then into (2.22) to give

$$\begin{aligned} \epsilon a' = \sigma\Bigg[&-a + \left\{\frac{\sigma + \tau}{\sigma(1-\tau)} + \mu\epsilon^2\right\}\left\{a - \epsilon a' + \epsilon^2(a'' - a^3) - \epsilon^3\left(a''' - \left(4 + \frac{2}{\varpi}\right)a^2 a'\right)\right\} \\ &- \left\{\frac{\tau^2(\sigma+1)}{\sigma(1-\tau)} + \epsilon^2\right\}\left\{\frac{a}{\tau} - \frac{\epsilon a'}{\tau^2} + \frac{\epsilon^2}{\tau^3}(a'' - \epsilon^3) - \frac{\epsilon^2}{\tau^4}\left(a''' - \left(4 + \frac{2}{\varpi}\right)a^2 a'\right)\right\}\Bigg] + O(\epsilon^4). \end{aligned} \tag{2.34}$$

It is easily verified that the $O(1)$ and $O(\epsilon)$ terms in this equation vanish identically, so that a factor ϵ^2 may be divided out leaving

$$a'' - a^3 + Na = \epsilon F(a) + O(\epsilon^2), \tag{2.35}$$

where

$$N = (1-\mu\tau)\,\sigma/\Delta, \quad \Delta = 1+\sigma+\tau, \tag{2.36}$$

and

$$F(a) = \left(1+\frac{1+\sigma}{\tau\Delta}\right)\left\{a''' - a'a^2\left(4+\frac{2}{\varpi}\right)\right\} + \frac{\sigma a'}{\tau\Delta}(1-\mu\tau^2). \tag{2.37}$$

Since a_3, b_3, and d_3 do not appear in (2.35), this equation gives the leading-order behaviour both of the full partial differential equations and of the 5th-order truncated system considered by Da Costa *et al.* (1981). Note that the order of the equations has been reduced to 3: the implications of this are discussed near the end of the section.

2.4. *The oscillatory solution*

For small ϵ we must have, at leading order,

$$a'' - a^3 + Na = 0. \tag{2.38}$$

If $N > 0$ (i.e. $r_T < r_T^{(e)}$) this equation has periodic solutions and can be solved in terms of a single parameter family of Jacobian elliptic functions. If the parameter $m[0 \leqslant m < 1]$ is defined in terms of the period Π of the oscillation by

$$\Pi = 4\left(\frac{1+m}{N}\right)^{\frac{1}{2}} K(m), \tag{2.39}$$

where $K(m)$ is the complete elliptic integral of the first kind, then we may write

$$a = \left(\frac{2mN}{1+m}\right)^{\frac{1}{2}} \operatorname{sn}\left(\left(\frac{N}{1+m}\right)^{\frac{1}{2}} t^* \middle| m\right). \tag{2.40}$$

(For details of the Jacobian elliptic functions and their integrals, see, for example, Davis (1962), but note that he uses k^2 where we use m.) It can then be easily shown that the maximum amplitude $a_{\max}$ and the root mean square amplitude a_{rms} are given by

$$\left.\begin{aligned} a_{\max} &= \left(\frac{2mN}{1+m}\right)^{\frac{1}{2}}, \\ a_{\text{rms}} &= \left\{\frac{1}{m}\left(1-\frac{E(m)}{K(m)}\right)\right\}^{\frac{1}{2}} a_{\max}, \end{aligned}\right\} \tag{2.41}$$

where $E(m)$ is the complete elliptic integral of the second kind. It remains to determine the parameter m that corresponds to a periodic solution for given μ. This may be done by the method of averaging (Jordan & Smith 1977). In this case the slowly varying quantity is the Hamiltonian $\mathscr{E}$, defined at leading order by

$$\mathscr{E} \equiv \tfrac{1}{2}a'^2 - \tfrac{1}{4}a^4 + \tfrac{1}{2}Na^2 = \frac{mN^2}{(1+m)^2}. \tag{2.42}$$

If we now multiply (2.35) by a' we obtain

$$\frac{d\mathscr{E}}{dt^*} = \epsilon a'\,F(a) + O(\epsilon^2), \tag{2.43}$$

so that $\mathscr{E}$ evolves on the slow time scale $T^* = \epsilon t^*$. The right-hand side varies on the time scale t^*, but only its average will affect $\mathscr{E}$. Thus we have, since $\mathscr{E}$ is independent of t^* to leading order,

$$\frac{d\mathscr{E}}{dT^*} = \langle a' F(a) \rangle, \tag{2.44}$$

where a is evaluated for $\epsilon = O$ and

$$\langle \ldots \rangle \equiv \Pi^{-1} \int_0^{\Pi} \ldots dt^*.$$

The integral on the right-hand side can be found in terms of N and m, using the properties of elliptic functions. After some algebra we find that (2.44) can be written in the form

$$\frac{d}{dT^*}\left[\frac{m}{(1+m)^2}\right] = -\left(1+\frac{1+\sigma}{\tau\Delta}\right)\left\{\left(1+\frac{2}{\varpi}\right) N I_1(m) + \left(N - \frac{\sigma(1-\mu\tau^2)}{1+\sigma+\tau\Delta}\right) J_1(m)\right\}, \tag{2.45}$$

where $I_1(m)$ and $J_1(m)$ are functions of m only, and are given in the appendix. Equation describes the evolution of m and hence $\mathscr{E}$ at fixed N or μ. The critical points of this equation give the periodic solutions of the problem. Equating the right-hand side to zero we obtain

$$\mu = \frac{1}{\tau}\left\{\frac{\alpha + \gamma f_1(m)}{1 + \gamma f_1(m)}\right\}, \tag{2.46}$$

where
$$\alpha = \frac{\tau(\sigma+\tau)}{1+\sigma}, \quad \gamma = \left(1 + \frac{\tau\Delta}{1+\sigma}\right)\left(1+\frac{2}{\varpi}\right), \tag{2.47}$$
and
$$f_1(m) = I_1(m)/J_1(m).$$

Equations (2.15), (2.40) and (2.46) determine parametrically the amplitude of the oscillations as a function of the Rayleigh number r_T. Since γ is always positive and $\alpha < 1$ it follows that $\mu\tau < 1$ for all m; moreover $f_1(m)$ is a monotonic function of m, rising from zero when $m = 0$ to 0.2 when $m = 1$. Figure 1 shows $a_{\max}$, a_{rms} and Π as functions of μ for $\sigma = 1$, $\tau = 1/8$ and $\varpi = 8/3$ (the last corresponds to cells with $\lambda = \sqrt{2}$ and gives the most unstable mode for $r_S \to 0$) and figure 2 (*a*) shows a as a function of t^* for a representative value of μ. For $m = 0$ we recover the linearized result: $\mu = \alpha/\tau$ and the amplitude is infinitesimal, in agreement with (2.14*a*).

Another check is provided by comparing (2.4*b*) for small values of m with modified perturbation theory: we have

$$f_1(m) = \tfrac{1}{2}m + O(m^2), \quad m \ll 1,$$

so that
$$\mu\tau = \alpha + \tfrac{1}{2}\gamma m(1-\alpha)$$

from (2.4). However, from (2.41) we have

$$a_{\max}^2 = \frac{2\sigma}{1+\sigma}(1-\tau)m + O(m^2);$$

so eliminating m between these two relations, we obtain

$$r_T = r_T^{(o)} + \epsilon^2 \frac{\Delta}{4\sigma\tau}\gamma a_{\max}^2 + O(\epsilon^4),$$

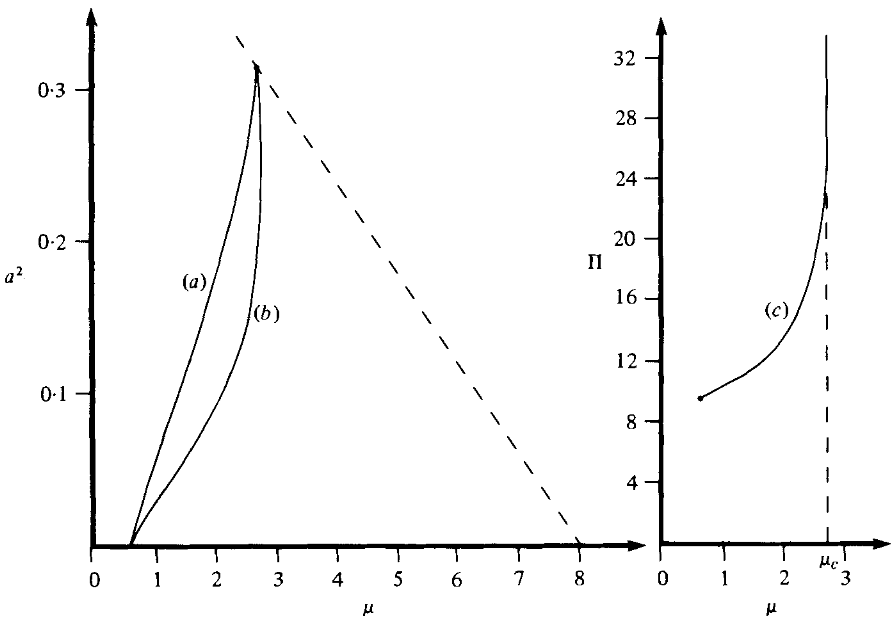

FIGURE 1. (a) a^2_{max}, (b) a^2_{rms}, (c) π (the period) for the oscillatory solution as functions of μ for the thermohaline problem when $\sigma = 1$, $\tau = 0{\cdot}125$ and $\varpi = \frac{8}{3}$. The dotted line shows the (unstable) steady solution branch. μ_c, the critical value of μ, is 2·674 in this case.

which is the same result as that obtained directly from the modified perturbation theory about $r_T^{(o)}$ (Huppert & Moore 1976) in the present limit. Note in particular that $r_T^{(o)}$ is always positive so that the bifurcation at $r_T^{(o)}$ is always supercritical. The Hopf bifurcation theorem (Hopf 1942) then shows that the branch of oscillatory solutions is stable in the neighbourhood of $r_T^{(o)}$.

For larger m, μ continues to increase monotonically. In the limit $m \to 1^-, f_1(m) \to \frac{1}{5}$, so that

$$\mu \to \mu_C = \frac{1}{\tau}\left\{\frac{5\alpha+\gamma}{5+\gamma}\right\}. \tag{2.48}$$

Then $a(t^*) \simeq \pm N^{\frac{1}{2}} \tanh(t^*\sqrt{N}/2)$ for $|t^*| < \Pi/4$, with the period Π approaching infinity as $-\ln(1-m)$. We conclude that $m = 1$ almost certainly represents a heteroclinic orbit that connects the two unstable steady branches (for negative and positive a) the latter are given in the limit by (2.38) as

$$a^2 = N = \sigma(1-\mu\tau)/\Delta, \tag{2.49}$$

and are also drawn in figure 1. It may be checked that these points indeed lie on the steady solution branch given by (2.11) in the present limit. These points of contact are then the 'critical points', where the limit cycle of the periodic solution intersects the steady solution branch. For $\mu > \mu_C$ there are no periodic solutions, at least in the parameter range under discussion. Figure 3 (from Da Costa *et al.*) shows a sketch of the topology of the phase diagram. No new effects are found by varying σ, τ or ϖ, so that the example shown is qualitatively correct for all parameter values.

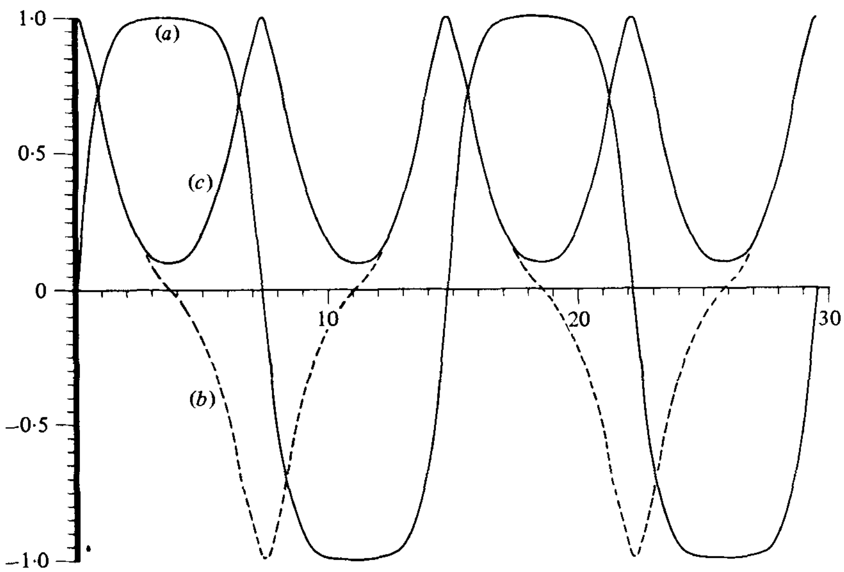

FIGURE 2. Graphs of (a) $\mathrm{sn}\,(x|0{\cdot}99)$, (b) $\mathrm{cn}\,(x|0{\cdot}99)$, (c) $\mathrm{dn}\,(x|0{\cdot}99)$, illustrating the different possible forms of the amplitude a as a function of t^* (a is snodial in the thermohaline problem, but in the magnetic problem the type of motion depends on the sign of the parameters M and N).

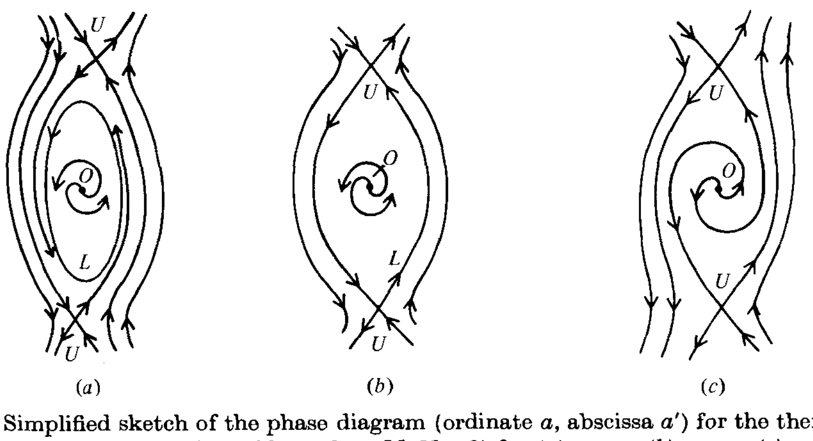

FIGURE 3. Simplified sketch of the phase diagram (ordinate a, abscissa a') for the thermohaline problem (and for the magnetic problem when $M, N > 0$) for (a) $\mu > \mu_c$, (b) $\mu > \mu_c$, (c) $\mu > \mu_c$. Note that no periodic solution is possible in (c) O is the unstable solution $a = 0$, L the stable limit cycle (when it exists), and U the unstable steady solutions of finite amplitude.

At this point it is necessary to remark on the possible non-uniformities in the expansion scheme. Firstly, it is clear that (2.35) represents a potential singular perturbation problem, since $F(a)$ contains higher derivatives of a than the left-hand side. Since the truncated system possesses periodic solutions, the corrections will remain small unless the period of the oscillation becomes very short; this does not however happen in this problem, or in the more involved magnetic problem considered in the next section. Secondly, for the averaging procedure to work the period must not be too *long*, for otherwise (2.44) would not be valid. Thus as $m \to 1$ in the calculation described above, $\Pi \sim \ln(1-m)$ and so the theory does not apply for $(1-m) \leqslant O(\exp(-\epsilon^{-1}))$.

It can be shown that if m is this close to unity, $\mu_c - \mu = O(t^{-1} \exp(-2\epsilon^{-1}))$ so for small ϵ the solution is valid everywhere except in an asymptotically small neighbourhood of μ_c. The third problem concerns the reduction of the equation from fifth order to two as a result of the expansion. While the equations certainly have a periodic solution in the neighbourhood of $r_T = r_T^{(0)}$, there remains the possibility that other solutions exist which differ from this periodic solution, but collapse on to it in the (a, a') phase plane in the small ϵ limit. We cannot investigate this possibility here; however, the close qualitative correspondence between our results and those of the full numerical problem suggest that even if there is a multiplicity of solutions the periodic solution we have found is the stable one.

2.5. *Orbital stability*

The periodic solutions we have found may or may not be unstable to small perturbations; thus we must examine their orbital stability. We therefore return to the relation (2.45) which shows how m changes when (2.46) is not satisfied. Setting $m = m_\mu + \delta m(T^*)$ where m_μ is that value of m for which (2.46) holds, we have at leading order

$$\frac{(1-m_\mu)}{(1+m_\mu)^3}\frac{d}{dT^*}(\delta m) = -\left(1+\frac{1+\sigma}{\tau\Delta}\right)\left\{\left(1+\frac{2}{\varpi}\right) NI_1'(m_\mu) + \left(N - \sigma\frac{(1-\mu\tau^2)}{1+\sigma+\tau\Delta}\right)J_1'(m_\mu)\right\}\delta m$$

$$= -J_1(m_\mu)\left(1+\frac{1+\sigma}{\tau\Delta}\right)\left(1+\frac{2}{\varpi}\right)Nf_1'(m_\mu)\,\delta m. \tag{2.50}$$

Thus $(1/\delta m)\, d(\delta m)/dT^*$ has the opposite sign to df_1/dm and hence since $df_1/dm > 0$ for all $m < 1$, stability is assured if $\mu < \mu_c$. For $\mu > \mu_c$ the periodic solution does not exist and since the steady branch is locally unstable, the solution must move out of the domain of the theory.

2.6. *Comparison with previous work*

The truncated equations (2.22)–(2.26) studied above represent a rational approximation to the full problem (2.2)–(2.4) for $\epsilon \ll 1$. However, Da Costa *et al.* showed by a combination of numerical and analytical techniques that their solutions are in good qualitative agreement with those obtained by Huppert & Moore (1976) for the full problem, even when $\epsilon = O(1)$. In particular, they found that provided the amplitude of oscillatory convection is moderate, it increases monotonically with the applied Rayleigh number, and that the oscillatory solution branch terminates on the unstable subcritical steady solution branch where the oscillation period becomes quite suddenly infinite. They interpreted this behaviour in terms of a limit cycle becoming singular and joining the two saddle points corresponding to the subcritical steady branch (figure 3). With r_T increased beyond the critical value, no oscillatory solutions were found and the solution jumped to a higher-amplitude (stable) portion of the steady branch. All these phenomena persist for smaller values of ϵ and are recovered here analytically; in particular the singular limit cycle and the value of r_T at which it forms are given by equation (2.48) in the present limit. This is strikingly demonstrated by comparing figures 1 and 2(*a*), of the present work with figures 1–4 of Da Costa *et al.* However the more exotic behaviour of the solution at large amplitudes found by these authors remains inaccessible analytically.

3. Convection in a vertical magnetic field

3.1. *Basic equations*

In this section we analyse the problem of two-dimensional convection in a horizontal Boussinesq layer of electrically conducting fluid in an imposed vertical magnetic field that is uniform and of magnitude $\mathbf{B}_0$ in the absence of motion. As we shall see, the linearized stability theory for this problem is almost identical to the thermohaline problem treated in § 2, but the presence of a quadratic restoring force in the equation of motion, coupled with the fact that the magnetic field tends to form boundary layers that are vertical rather than horizontal, means that a far greater variety of behaviour is possible when nonlinear effects are taken into account. The full range of phenomena is strikingly demonstrated in the numerical work of Weiss (1981*a*, *b*) and the analysis of a truncated modal system (showing the same bifurcation properties) by Knobloch *et al.* (1981).

If effects of magnetic buoyancy are neglected, then the Boussinesq equation of state is $\rho = \rho_0(1-\alpha(T-T_0))$. The dimensionless governing equations are then conveniently written in terms of the fluctuation temperature Θ (equation (2.4)), the stream function ψ (equation (2.2)) and a flux function $A(x, z)$ where $\mathbf{B}$, the magnetic intensity, is defined by

$$\mathbf{B} = |\mathbf{B}_0|\{\hat{\mathbf{z}} + (-\partial_z A, O, \partial_x A)\}. \tag{3.1}$$

These equations take the form (Weiss 1977)

$$\sigma^{-1}[\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi)] = R_T \partial_x \Theta + \zeta Q J(x+A, \nabla^2 A) + \nabla^4 \psi, \tag{3.2}$$

$$\partial_t \Theta + J(\psi, \Theta) = \partial_x \psi + \nabla^2 \Theta, \tag{3.3}$$

$$\partial_t A + J(\psi, A) = \partial_z \psi + \zeta \nabla^2 A, \tag{3.4}$$

where σ and R_T are given by (2.6), and ζ and Q are defined by

$$\zeta = \frac{\eta}{\kappa_T}, \quad Q = \frac{B_0^2 h^2}{\mu_0 \rho_0 \eta \nu}. \tag{3.5}$$

Here η is the magnetic diffusivity of the fluid, and μ_0 its permeability. The boundary conditions on ψ and Θ are the same as those of § 2, while $A(x, z)$ satisfies

$$A = 0, \quad x = 0, \lambda; \quad \partial_z A = 0, \quad z = 0, 1. \tag{3.6}$$

Thus the total flux in each cell is fixed, but the field lines are free to move horizontally along the upper and lower boundaries, though they must remain vertical there.

As in the thermohaline problem, the system has the static solution $\psi = \Theta = A = 0$. The linearized stability theory can be simply stated by using notation similar to that of § 2. Writing

$$r_T = \frac{\pi^2}{\lambda^2 p^3} R_T, \quad q = \frac{\pi^2}{p^2} Q, \quad p = \pi^2(1+\lambda^{-2}), \tag{3.7}$$

and $\Delta = 1+\sigma+\zeta$, the results are identical to those of the thermohaline problem with r_S replaced by ζ_q and τ replaced by ζ. Thus we have (Weiss 1964; Knobloch *et al.* 1981):

(*a*) if
$$\zeta > 1 \quad \text{or} \quad q < q_c = \frac{\zeta(\sigma+1)}{\sigma(1-\xi)} \tag{3.8}$$
instability sets in as a direct mode at
$$r_T \equiv r_T^{(e)} = 1+q,$$
(*b*) if
$$\zeta < 1 \quad \text{and} \quad q > q_c$$
instability sets in as an oscillatory mode at
$$r_T \equiv r_T^{(o)} = 1+\frac{\Delta\zeta}{\sigma}+\zeta\left(\frac{\sigma+\zeta}{\sigma+1}\right)q. \tag{3.9}$$

We shall in what follows suppose that case (ii) applies. As in the thermohaline case, a branch of finite amplitude oscillatory solutions bifurcates from $r_T^{(o)}$, and a branch of finite-amplitude steady solutions from $r_T^{(e)}$. We wish to study how the oscillatory solution changes with position on the branch. In the present case, unlike the thermohaline problem, parts of the branch are typically unstable, so that they cannot be followed by numerical means. Analytical results are available only when q is close to q_c and the period of the oscillations is long. Even in this limited range, the variety of behaviour is very wide. Depending on the parameters, both the steady and oscillatory branches can be either subcritical or supercritical. If the steady branch is subcritical, the oscillatory branch is supercritical and stable, and ends on the steady branch via a heteroclinic orbit as in the thermohaline case. If both branches are supercritical, the oscillatory solution is 'cnoidal' rather than 'snoidal' in form; it eventually loses stability and then turns into a 'dnoidal' oscillation with non-zero mean (see figure 2) which eventually joins the steady branch via a Hopf bifurcation. Above this bifurcation the steady branch is stable. There are other, less interesting possibilities, also.

As in the previous section, we suppose $O < q-q_c \ll 1$ and make an expansion in powers of $(q-q_c)^{\frac{1}{2}}$. Again we shall find it convenient to write down a modal representation of the equations, and apply the same methods as before to obtain a solution.

3.2. *The problem in the limit $q \to q_c$*

We set
$$q = q_c+\epsilon^2, \quad \epsilon \ll 1, \tag{3.10}$$
and then from (3.8), (3.9) find that
$$\left.\begin{aligned} r_T^{(o)} &= \frac{\sigma+\zeta}{\sigma(1-\zeta)}+\frac{\zeta(\sigma+\zeta)}{(\sigma+1)}\epsilon^2, \\ r_T^{(e)} &= \frac{\sigma+\zeta}{\sigma(1-\zeta)}+\epsilon^2. \end{aligned}\right\} \tag{3.11}$$
We thus set
$$r_T = \frac{\sigma+\zeta}{\sigma(1-\zeta)}+\mu\epsilon^2, \tag{3.12}$$
where $\mu = O(1)$. As before, the amplitude of the motion is of order ϵ in this case, and the frequency of linear oscillations is also of order ϵ. We therefore define
$$t^* = \epsilon p t \tag{3.13}$$

as a new time scale. Following the methods that lead to (2.19)–(2.21), we write down a modal representation of ψ, Θ, A:

$$\psi = 2(2p)^{\frac{1}{2}}\frac{\lambda}{\pi}\{\epsilon\sin(\pi\chi/\lambda)\sin\pi z\, a_{11}(t^*) + \epsilon^3\sin(3\pi\chi/\lambda)\sin\pi z\, a_{31}(t^*)$$

$$+ \epsilon^3\sin(\pi\chi/\lambda)\sin 3\pi z\, a_{13}(t^*) + \ldots, \tag{3.14}$$

$$\Theta = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\epsilon\cos(\pi\chi/\lambda)\sin\pi z\, b_{11}(t^*) - \frac{1}{\pi}\epsilon^2\sin 2\pi z\, c(t^*)$$

$$+2\left(\frac{2}{p}\right)^{\frac{1}{2}}\epsilon^3\{\cos(3\pi\chi/\lambda)\sin\pi z\, b_{31}(t^*) + \cos(\pi\chi/\lambda)\sin 3\pi z\, b_{13}(t^*)\} + \ldots, \tag{3.15}$$

$$A = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\lambda\epsilon\sin(\pi\chi/\lambda)\cos\pi z\, d_{11}(t^*) + \frac{\lambda}{\pi}\epsilon^2\sin(2\pi\chi/\lambda)\, e(t^*)$$

$$+2\left(\frac{2}{p}\right)^{\frac{1}{2}}\lambda\epsilon^3\{\sin(3\pi\chi/\lambda)\cos\pi z\, d_{31}(t^*) + \sin(\pi\chi/\lambda)\cos 3\pi z\, d_{13}(t^*)\} + \ldots. \tag{3.16}$$

Then substituting into (3.2)–(3.4), we obtain a set of modal equations of which the first five are (cf. Knobloch *et al.* 1981)

$$\epsilon a'_{11} = \sigma\left[-a_{11} + b_{11}\left\{\frac{\sigma+\zeta}{\sigma(1-\zeta)} + \mu\epsilon^2\right\} - d_{11}\left\{\frac{\zeta^2(\sigma+1)}{\sigma(1-\zeta)} + \zeta\epsilon^2\right\}(1 + (3-\varpi)\epsilon^2 l)\right] + O(\epsilon^4), \tag{3.17}$$

$$\epsilon b'_{11} = -b_{11} + a_{11} - \epsilon^2 a_{11}c + O(\epsilon^4), \tag{3.18}$$

$$\epsilon c' = \varpi(-c + a_{11}b_{11}) + O(\epsilon^2), \tag{3.19}$$

$$\epsilon d'_{11} = -\zeta d_{11} + a_{11} - \epsilon^2 a_{11}e + O(\epsilon^4), \tag{3.20}$$

$$\epsilon e' = -(4-\varpi)\zeta e + \varpi a_{11}d_{11} + O(\epsilon^2), \tag{3.21}$$

where the prime again denotes differentiation with respect to t^*. The chief difference between this and the thermohaline problem is the different appearance of ϖ in (3.21) than in (2.26) and the nonlinear term in (3.17) that arises from the Lorenz force. As before $O < \varpi < \psi$. This problem can be solved sequentially in powers of ϵ just as the thermohaline problem was. We therefore omit the details and move on directly to the consideration of the resulting simplified system.

3.3. *Solution of the reduced equation*

After some algebra we finally obtain an equation for a_{11} (with the subscripts dropped for convenience) of the form

$$a'' - Ma^3 + MNa = \epsilon F(a), \tag{3.22}$$

with

$$M = -r_2^{(e)}\sigma\zeta/\Delta, \quad N = -(1-\mu)/r_2^{(e)} \tag{3.23}$$

and

$$F(a) = \sigma[Ca^2a' + Da''' + (1-\mu\zeta)a']/\Delta,$$

where

$$\left.\begin{aligned} C &= \frac{2\zeta(\zeta+\sigma)}{\sigma(1-\zeta)}\left(2+\frac{1}{\varpi}\right) - \frac{(1+\sigma)}{\zeta\sigma(1-\zeta)}\left(\frac{2\varpi}{4-\varpi}\right)\left[\varpi - 1 + \frac{\varpi-2}{4-\varpi}\right], \\ D &= 1+\frac{\zeta}{\sigma}+\frac{\Delta}{\zeta\sigma}. \end{aligned}\right\} \tag{3.24}$$

It will be noted that we have used some of the same symbols as in §2, so as to make comparison simpler.

The quantity $r_2^{(e)}$ appearing in (3.23) is defined by

$$r_2^{(e)} = -\frac{(1+\sigma)}{\zeta\sigma(1-\zeta)}\left[\frac{\varpi(\varpi-2)}{4-\varpi} - \alpha\right]$$

where

$$\alpha = \frac{\zeta(\sigma+\zeta)}{1+\sigma}. \tag{3.25}$$

The significance of the parameter $r_2^{(e)}$ can be seen if we investigate the *steady* solutions of (3.22) (for which $F(a) = 0$). Clearly

$$a^2 = N$$

so that

$$\mu = 1 + r_2^{(e)}\, a^2. \tag{3.26}$$

The same result follows from modified perturbation theory applied to the full equations in the present limit. Thus if $r_2^{(e)} > O$, the steady branch is supercritical, and if $r_2^{(e)} < 0$ it is subcritical. It is clear that the sign of $r_2^{(e)}$ has an important effect on the solution of (3.22). Before we can proceed to enumerate the different cases that arise, we should obtain the energy evolution equation. As before, we define

$$\mathscr{E} = \tfrac{1}{2}a'^2 - \frac{M}{4}a^4 + \frac{MN}{2}a^2,$$

so that

$$d\mathscr{E}/dT^* = \langle a'F(a)\rangle, \quad T^* = \epsilon t^*. \tag{3.27}$$

By using the leading-order form for a, we obtain

$$d\mathscr{E}/dT^* = \sigma[\langle a^2a'^2\rangle(C+3MD) + \langle a'^2\rangle(1-\mu\zeta - DMN)]/\Delta. \tag{3.28}$$

It will emerge that both terms on the right-hand side can take either sign, and do not depend monotonically on $\mathscr{E}$, so that stability is no longer assured. In order to evaluate the terms in brackets we have to solve (3.22) with $\epsilon = 0$. The solution depends on the sign of $r^{(e)}$.

3.4. *Solution in the case $r_2^{(e)} < 0$ (subcritical steady solution)*

In this case M and N are both positive and (3.22) has the same form as (2.35) (μ cannot be greater than unity since otherwise the solution is not periodic). In fact, if the period is given by

$$\Pi = 4\left(\frac{1+m}{MN}\right)^{\frac{1}{2}} K(m), \tag{3.29}$$

then

$$\left.\begin{aligned} a &= \left(\frac{2mN}{1+m}\right)^{\frac{1}{2}} \operatorname{sn}\left(\left(\frac{MN}{1+m}\right)^{\frac{1}{2}} t^* \,|\, m,\right) \\ a_{\max} &= \left(\frac{2mN}{1+m}\right)^{\frac{1}{2}}, \\ a_{\mathrm{rms}} &= \left\{\frac{1}{m}\left[1-\frac{E(m)}{K(m)}\right]\right\}^{\frac{1}{2}} a_{\max}, \\ \text{and} \qquad \mathscr{E} &= \frac{mMN^2}{(1+m)^2}. \end{aligned}\right\} \tag{3.30}$$

Substitution in (3.28) then yields

$$\left.\begin{aligned} \frac{d}{dT^*}\left[\frac{m}{(1+m)^2}\right] &= \frac{\sigma}{\Delta}\left[\frac{1-\mu}{|r_2^{(e)}|}\left(C-3Dr_2^{(e)}\frac{\sigma\tau}{\Delta}\right)I_1(m)+\left(1-\mu\zeta-\frac{D\sigma\zeta}{\Delta}(1-\mu)\right)J_1(m)\right] \\ &= \frac{\sigma\beta}{\Delta}[\mu(J_1+\gamma I_1)-(\alpha J_1+\gamma I_1)] \end{aligned}\right\} \tag{3.31}$$

where $\beta = (1+\sigma)/\Delta, I_1(m)$ and $J_1(m)$ are defined as before, and

$$\gamma = -\left\{3(1+\alpha\beta)\left[\frac{\varpi(\varpi-2)}{4-\varpi}-\alpha\right]-2\alpha(2+\varpi^{-1})\zeta - \frac{2\omega}{4-\omega}(\varpi-1+(\varpi-2)/(4-\varpi))\right\}\Big/\left(\beta\left|\frac{\varpi(\varpi-2)}{4-\varpi}-\alpha\right|\right). \tag{3.33}$$

Thus we have, for the periodic solutions,

$$\mu = \frac{\alpha+\gamma f_1(m)}{1+\gamma f_1(m)}, \tag{3.33}$$

where $f_1(m)$ is defined as in §2. As before, for small m, we find

$$\left.\begin{aligned} r_T &= r_T^{(o)} + \epsilon^2 r_2^{(o)} + O(\epsilon^4) \\ \text{where} \qquad r_2^{(o)} &= \tfrac{1}{4}\gamma\,|\,r_2^{(e)}\,|\,a_{\max}^2, \end{aligned}\right\} \tag{3.34}$$

in agreement with the modified perturbation result derived by Knobloch *et al.* Hence $r_2^{(o)}$ has the same sign as γ, and in the present case when $r_2^{(e)} < 0, \gamma$ is always positive, as we shall show below. Thus the bifurcation is supercritical, and the oscillatory branch meets the unstable steady branch at $\mu = \mu_c = (5\alpha+\gamma)/(5+\gamma)$. The branch is stable throughout, just as in the thermohaline problem, although the same reservations apply about the validity of the scheme when $\mu \simeq \mu_c$. A typical solution is shown in figure 4. The case $r_2^{(e)} > 0$ is much more complicated, and we consider it next. It will be found helpful for what follows to refer to figure 5.

3.5. *Solution in the case $r_2^{(e)} > 0$ (supercritical steady solution)*

This case, like Gaul (Caesar 51 B.C.), must be considered in three parts. To begin with, we suppose that we are in the neighbourhood of $r_T^{(o)}$ so that $\mu < 1$. Then $M < 0, N < 0$.

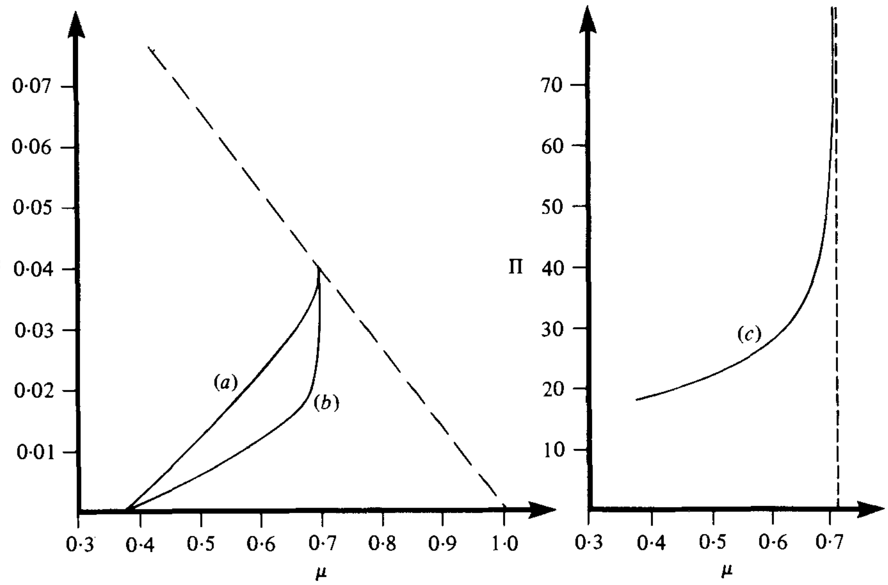

FIGURE 4. As for figure 1, but for the magnetic problem in a case where the steady solution branch is subcritical (case (i) of §3·6). Particular parameters are: $\varpi = 8/3$, $\sigma = 1$, $\zeta = 0{\cdot}5$; $r_2^{(e)} = -7{\cdot}667$, $\alpha = 0{\cdot}375$, $\gamma = 5{\cdot}267$.

The equation (3.22) is then solved parametrically by

$$\left.\begin{aligned} \Pi &= 4\left(\frac{1-2m}{MN}\right)^{\frac{1}{2}} K(m) \quad (m < \tfrac{1}{2}), \\ a &= \left(-\frac{2mN}{1-2m}\right)^{\frac{1}{2}} \operatorname{cn}\left(\left(\frac{MN}{1-2m}\right)^{\frac{1}{2}} t^* \,|\, m\right), \\ a_{\max} &= \left(-\frac{2mN}{1-2m}\right)^{\frac{1}{2}}, \quad a_{\text{rms}} = \left[1-\frac{1}{m}\left(1-\frac{E(m)}{K(m)}\right)\right]^{\frac{1}{2}} a_{\max}, \\ \mathscr{E} &= -MN^2\frac{m(1-m)}{(1-2m)^2}. \end{aligned}\right\} \tag{3.35}$$

Substitution into (3.28) then gives

$$\frac{d}{dT^*}\left[\frac{m(1-m)}{(1-2m)^2}\right] = \frac{\sigma\beta}{\Delta}[\mu(J_2+\gamma I_2)-(\alpha J_2+\gamma I_2)] \tag{3.36}$$

where $I_2(m)$, $J_2(m)$ are given in the appendix, and γ is the *same* as in (3.32) (but note the modulus sign in the denominator of that expression). Thus the value of μ for periodic solutions is given by

$$\mu = \frac{\alpha+\gamma f_2(m)}{1+\gamma f_2(m)}; \quad f_2(m) = I_2(m)/J_2(m). \tag{3.37}$$

This regime corresponds to the segments AS, AS' in figure 5.

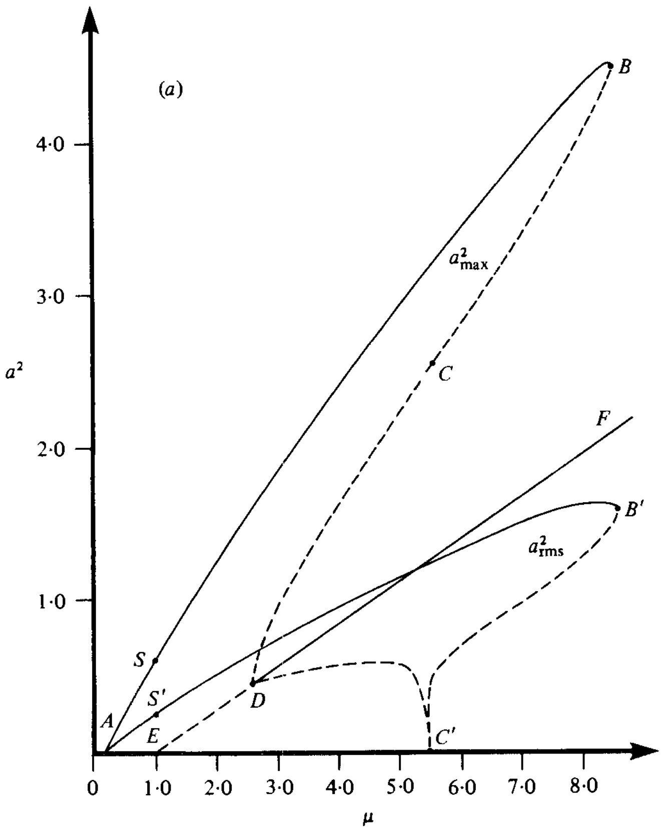

FIGURE 5 (a). For legend see opposite.

For m close to zero, the results of modified perturbation theory are recovered as before. As $m \to \frac{1}{2}^-$, $f_2(m) \to \infty$ and $\mu \to 1^-$. $f_2(m)$ increases monotonically if $m \propto \frac{1}{2}$, so that if $\gamma > 0$, $d\mu/dm > 0$ and it may be verified from (3.36) that the solution is stable in this case. If on the other hand $\gamma < 0$, then $\mu \leqslant \alpha$ for all m which yield a value of μ that is less than 1. The solution is then subcritical and unstable.

Although there is a singularity in the representation at $m = \frac{1}{2}$, $\mu = 1$, there is no singularity in the solution there and in fact, as might be expected, the cnoidal solution branch passes smoothly through $\mu = 1$.

If we now turn to the solutions which obtain for $\mu > 1$, we note that $M < 0$ and $N > 0$. Then since also $m > \frac{1}{2}$ the solution (3.35) still holds for the cnoidal modes, as the quantities under square root signs remain positive. Substitution into (3.28) then yields

$$\frac{d}{dT^*}\left[\frac{m(1-m)}{(1-2m)^2}\right] = \frac{\sigma\beta}{\Delta}[\mu(J_2-\gamma I_2)-(\alpha J_2-\gamma I_2)] \tag{3.38}$$

so that periodic solutions satisfy

$$\mu = \frac{\gamma f_2(m)-\alpha}{\gamma f_2(m)-1}, \tag{3.39}$$

corresponding to the segments SBC, $S'B'C'$ in the figure. This solution is only valid if $\gamma > 0$. Thus $d\mu/dm$ has the opposite sign to df_2/dm, and near $m = \frac{1}{2}^+$, $df_2/dm < 0$ so

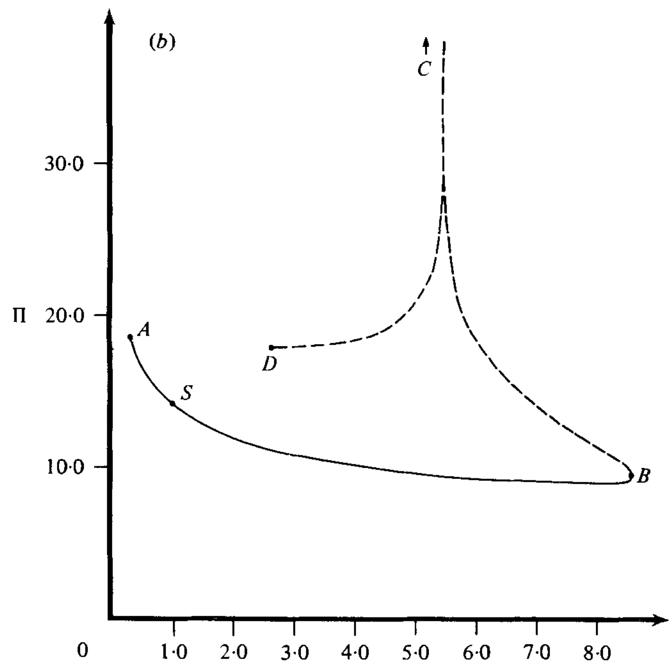

FIGURE 5. (a) a^2_{max} (curve $ASBCD$) and a^2_{rms} (curve $AS'B'C'D$) for the magnetic problem in case (ii) of §3·6 ($r_2^{(e)} > 0$, $\gamma > 0$) as a function of μ. The straight line EDF is the steady solution branch. Dotted and full lines denote unstable and stable solutions, respectively. $AB(AB')$ stable cnoidal solution; $BC(B'C')$ unstable cnoidal solution; $CD(C'D)$ unstable dnoidal solution. (b) The period Π of the above cnoidal oscillations and 2Π for the dnoidal oscillations (segment CD). Parameters are: $\varpi = 0{\cdot}4$, $\sigma = 1$, $\zeta = 0{\cdot}35$; $r_2^{(e)} = 3{\cdot}64$, $\alpha = 0{\cdot}236$, $\gamma = 1{\cdot}463$.

that μ increases with m. However, f_2 reaches a minimum as a function of m at $m = 0.93$, where it takes the value $\delta = 0.752$, and then increases to 0.8 as $m \to 1^-$. Investigation of (3.38) shows that the solution is stable if and only if $df_2/dm < 0$.

Hence the cnoidal solution that bifurcates at $\mu = \gamma$ loses stability at $\mu = \mu_\delta = (\gamma\delta - \alpha)/(\gamma\delta - 1)$, at which point μ has a maximum as a function of a_{max} (see figure 5). If $\gamma < 1/\delta$ then μ_δ is 'off scale' and so the oscillatory branch goes 'to infinity' without losing stability; a higher-order expansion is necessary to determine where the maximum of μ occurs. As m is further increased μ and a_{max} both descend and at $m \simeq 1$, $\mu = (4\gamma - 5\alpha)/(4\gamma - 5)$, the period tends to infinity and the (unstable) solution then takes the form

$$a = \pm (2N^{\frac{1}{2}}) \operatorname{sech} [(-MN)^{\frac{1}{2}} t^*]. \tag{3.40}$$

The situation becomes clearer if one imagines a to represent the co-ordinate of a particle in a potential well (the general shape of which, for $r_2^{(e)} > 0$, $\mu > 1$ is shown in figure 6). The cnoidal oscillation has zero mean, and is represented by a particle of positive energy. As μ changes both the shape of the well and the energy of the particle change, and as $m \to 1^-$ energy of the cnoidal oscillation tends to zero and the particle can only just reach the origin.

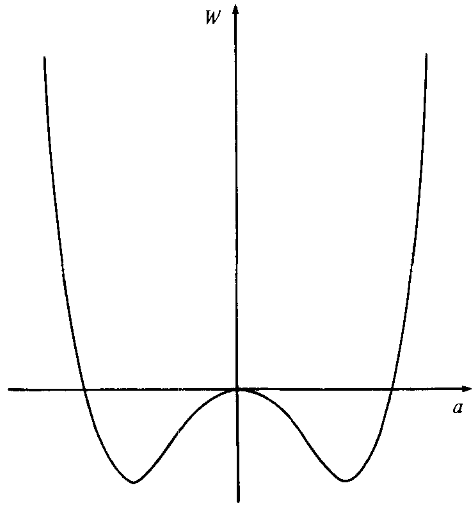

FIGURE 6. Sketch of the potential well $W(a)$ that controls the form of $a(t^*)$ in the magnetic problems when $r_2^{(r)} > 0$, $\mu > 1$. It may be seen that the oscillation can be of either cnoidol (zero-mean), or dnoidal (non-zero mean type), depending on its energy.

If the particle energy is negative oscillations are still possible but now, with non-zero mean, representing motion about the steady finite-amplitude solution given by the minimum of the potential for either sign of a. These are the 'dnoidal' solutions which we discuss next. Note first, however, that if $\gamma < 1.25$ the transition point between cnoidal and dnoidal motion is 'off scale' and cannot be found by the theory.

Solutions to (3.22) with non-zero mean, and $M < 0$, $N > 0$ as above, may be written in the form

$$\left.\begin{aligned} \Pi &= 2\left(\frac{m-2}{MN}\right)^{\frac{1}{2}} K(m), \\ a &= \left(\frac{2N}{2-m}\right)^{\frac{1}{2}} \mathrm{dn}\left(\left(\frac{MN}{m-2}\right)^{\frac{1}{2}} t^* \,\middle|\, m\right), \\ a_{\mathrm{max}} &= \left(\frac{2N}{2-m}\right)^{\frac{1}{2}}, \quad a_{\mathrm{rms}} = \left\{\frac{E(m)}{K(m)}\right\}^{\frac{1}{2}} a_{\mathrm{max}}, \\ \mathscr{E} &= MN^2\frac{(1-m)}{(2-m)^2}. \end{aligned}\right\} \tag{3.41}$$

Note that $\mathscr{E} < 0$. When m is small, a^2 is almost N; i.e. the solution represents small oscillations about the steady branch. When $m \simeq 1$, the period to infinity and the solution then tends to the same limiting form as the cnoidal branch (except, of course, that the ratio of the two periods tends to 2 as the dnoidal oscillation takes half as long to be traced out).

By analogy with previous cases, we can show that

$$\mu = \frac{\gamma f_3(m) - \alpha}{\gamma f_3(m) - 1}, \tag{3.42}$$

where $f_3(m) = I_3(m)/J_3(m)$ (see the appendix), and that the solution is stable or unstable according to whether $df_3(m)/dm$ is positive or negative. In fact $f_3(m)$ is a slowly decreasing function of m, being equal to 1 at $m = 0$ and $\frac{4}{5}$ at $m = 1$, and $df_3(m)/dm$ is zero only at $m = 0$, and negative elsewhere. Thus the dnoidal branch is unstable, except possibly in a neighbourhood of $m = 0$. If $m \lesssim O(\epsilon)$, the slope of $a(\mu)$ is determined by higher-order terms in the expansion, so that the bifurcation may be stable or unstable locally. This part of the solution corresponds to the segments CD, $C'D'$ in figure 5.

The value $\mu = (\gamma-\alpha)/(\gamma-1)$ is of particular interest, since it represents a point where the steady solution can coexist with infinitesimal oscillations. It is thus a Hopf bifurcation point, and marks the place where the steady solution branch, unstable near $a = 0$, gains stability to small perturbations as a increases. This fact provides a check on the analysis, since the truncated modal expansion has similar behaviour even when $\epsilon = O(1)$. The condition that the dispersion relation, characterizing the growth rate of a small disturbance from the steady solution, possesses two complex conjugate imaginary roots (the condition for a Hopf bifurcation) has been given by Knobloch *et al.* (1981). It can be shown, after much algebra, that the condition reduces to $\mu = (\gamma-\alpha)/(\gamma-1)$ in the present limit, as required.

As previously mentioned, the leading-order terms in the expansion give no information about the stability of the dnoidal solution in the neighbourhood of the Hopf bifurcation; the determination of the exact nature of the bifurcation when $\epsilon \ll 1$ would require a higher-order calculation. However, Knobloch *et al.* (1981) have made studies of the case $\epsilon = O(1)$, and find what appear to be subcritical (unstable) oscillations about the steady branch. While the modal equations show the property convincingly, the full two-dimensional problem studied by Weiss (1981*a,b*) apparently exhibits supercritical (stable) oscillations, although these may be unstable oscillations with a small positive growth rate.

The other interesting feature of the solution when $r_2^{(e)} > 0$, the transition from oscillations with zero mean to ones with non-zero mean, cannot be observed in a numerical study since the change takes place in the unstable region of the branch. However, the 'doubling back' of the oscillatory branch is observed in the numerical experiments, and is probably necessary for the termination of the oscillatory branch at a Hopf bifurcation. The results presented here reinforce the belief that even when ϵ is not small the oscillations with non-zero mean join those with zero mean in a straightforward manner.

3.6. *Classification of the solutions*

It is clear from what has gone before that the variety of solutions in the magnetic problem, though large, depends essentially on two parameters, $r_2^{(e)}$ and γ. The different cases that can arise in our limit are summarized as follows:

(i) $r_2^{(e)} < 0$; then $\gamma > 0$, the solution is snoidal and meets the subcritical steady branch in a heteroclinic orbit, without losing stability.

(ii) $r_2^{(e)} > 0$, $\gamma > \delta^{-1} = 1{\cdot}329$; the branch begins as a stable cnoidal oscillation, which loses stability at $\mu = (\gamma\delta-\alpha)/(\gamma\delta-1)$, changes form at $\mu = (4\gamma-5\alpha)/(4\gamma-5)$ to a dnoidal oscillation, and ends at $\mu = (\gamma-\alpha)/(\gamma-1)$ in a Hopf bifurcation on the supercritical steady branch.

(iii) $r_2^{(e)} > 0$, $\delta^{-1} > \gamma > 1{\cdot}25$. The stable cnoidal branch disappears 'off scale'; an

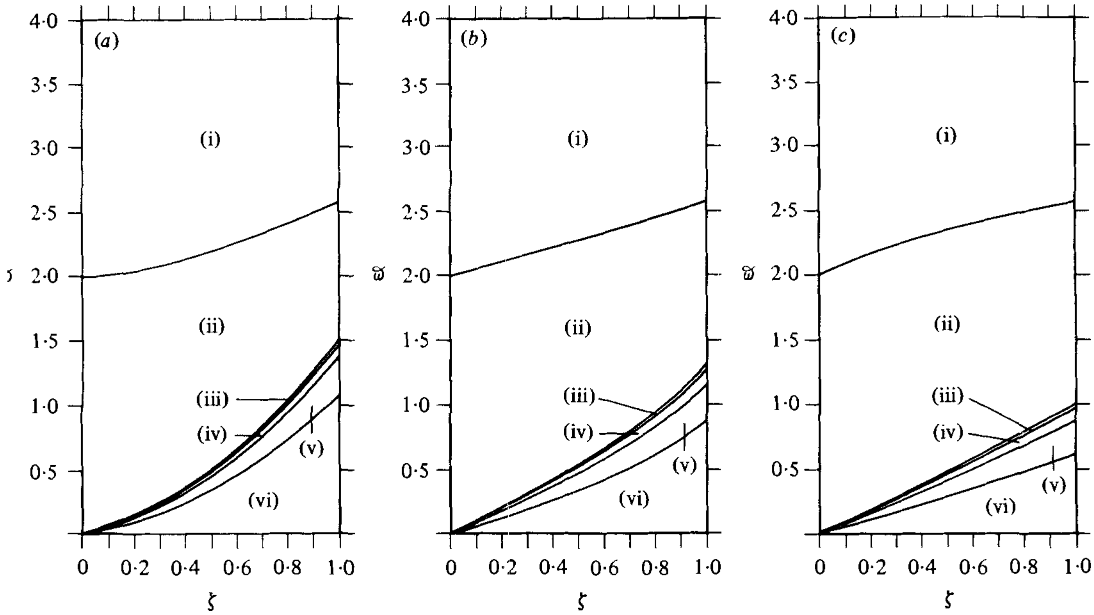

FIGURE 7. Location in (ζ, ϖ) space of the six regions describe in § 3·6 for (a) $\sigma = 10^{-5}$, (b) $\sigma = 1$, (c) $\sigma = 10^5$.

unstable cnoidal branch appears from infinity, becomes dnoidal and ends in a Hopf bifurcation.

(iv) $r_2^{(e)} > 0$, $1{\cdot}25 > \gamma > 1$. The stable cnoidal branch disappears 'off scale; an unstable dnoidal branch appears from infinity and ends in a Hopf bifurcation.

(v) $r_2^{(e)} > 0$, $1 > \gamma > 0$. The stable cnoidal branch disappears 'off scale'; there is no Hopf bifurcation within the limits of the theory.

(vi) $r_2^{(e)} > 0$, $\gamma < 0$. The cnoidal branch is subcritical and unstable and does not meet the steady branch (within the regime of validity of the theory).

Figure 7 shows the regions of (ζ, ϖ) space corresponding to the various cases when (a) $\sigma = 10^{-5}$, (b) $\sigma = 1$, (c) $\sigma = 10^5$. It will be seen that there is little qualitative difference between small and large σ in terms of the relative sizes of the various regions.

The truncated equations (3.17)–(3.21) provide a rational approximation to the full partial differential equations (3.2)–(3.4) when ϵ is small. However, the structure of the solutions and their stability properties are in excellent qualitative agreement with the numerical solutions both to the full two-dimensional problem (Weiss 1981 *a, b*) and to the truncated modal system studied by Knobloch *et al.* (1981). The stable portions of the numerical solutions agree well in their morphology with those found here, and it is entirely plausible to suppose that they are connected by an unstable regime of the kind found in the present paper. The only major question that is not resolved by the theory at leading order is the supercriticality of the Hopf bifurcation, as previously discussed. The behaviour of the various bifurcations etc. can be conveniently shown in a phase diagram as in figure 8. Similar bifurcation diagrams have been obtained by Takens (1974) and Carr (1979) in generic studies of oscillatory instability.

The general remarks on the validity of the analysis made at the end of § 2 apply with equal force to the calculations of this section. One further difficulty arises in the mag-

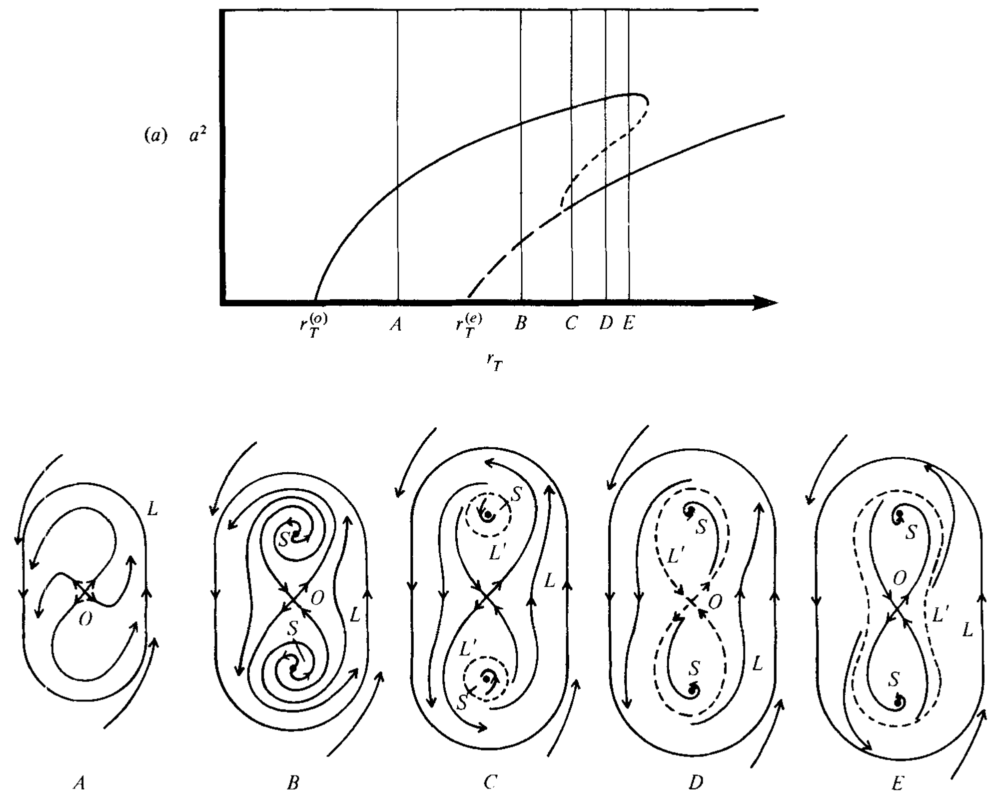

FIGURE 8. Sketch of the phase portraits for the magnetic problem when $r_2^{(e)} > 0$, $r_2^{(o)} > 0$. (*a*) Amplitude–Rayleigh-number diagram, showing the five regions of interest. (*b*) Phase portraits. Case A: The origin O is an unstable node or focus. Case B: O is now a saddle point, and the new singular points S, S' corresponding to steady non-zero solutions are either unstable nodes or foci. Case C: S, S' become stable, and unstable limit cycles (dashed) appear around them. Case D: Unstable heteroclinic orbits of infinite period form. Case E: Two 'concentric' limit cycles surround all three fixed points, the inner being unstable. The vertical and horizontal axes are a and a' throughout.

netic case: if $r_2^{(e)}$ is very small ($\leqslant O(\epsilon)$) then the question whether the steady solution branch is subcritical or supercritical is resolved by higher-order terms. The periodic solutions will then be sinusoidal in form except near $\mu = 1$, where these higher-order terms become important. In this case the amplitude equation will contain a quintic nonlinearity (cf. Rubenfeld & Siegmann 1977). The necessary calculations, which would be very involved, are outside the scope of the present paper.

4. Conclusions

In previous sections we have treated two well-known double-diffusive convective stability problems in a regime in which the Rayleigh numbers for the onset of overstable oscillations and for the onset of direct modes differ by only a small amount. In this regime the frequency of the oscillations is low, and the leading-order approximation to the equations governing the time development of the amplitude of the convection is that of a nonlinear oscillator. Higher-order terms fix the amplitude of the oscillation as a function of the Rayleigh number. For the thermohaline problem the steady branch of solutions is always subcritical and the oscillatory branch is supercritical and stable, and meets the branch in a heteroclinic orbit without losing stability. The magneto-convection problem has much more variety; the oscillations may be stable or unstable, and meet the steady branch either in a heteroclinic orbit or in a Hopf bifurcation depending on the physical parameters of the problem. All the properties can be explicitly demonstrated analytically, and they are in excellent qualitative agreement with numerical computations carried out for the full system of equations, in regions where agreement is to be expected. Analytical methods allow the solution to be found even when it is unstable, and we have been able to show that there is only one branch of oscillatory solutions, which changes its character but evolves continuously as the parameters change.

There are, of course, other double-diffusive problems which can be treated by this method. Foremost among them is that of convection in a uniformly rotating layer, for which overstability is possible if the Prandtl number $\sigma < 1$ (Veronis 1959, 1966, 1968*a*). The complexity of this problem is midway between that of the thermohaline and magnetic ones. We have investigated the solutions and found no qualitative differences between this problem and those discussed in the present paper; we therefore do not present the results at this time.

The present work is, to our knowledge, the first to show explicitly how oscillatory solutions evolve and change at finite amplitude. It is restricted in that only a particular planform of convection (two-dimensional rolls in this case) can be considered. We are currently considering extensions of the theory, since one of the prime objectives of any non-linear analysis must be to determine the planform that actually evolves from a given set of initial disturbances. One limited step is to consider those values of ϖ for which the layer loses stability to two different modes at the same value of R_T, but calculations on this are as yet at a preliminary stage.

We are most grateful to A. M. Soward for communicating to us his unpublished work on the magnetic problem, in which he presents an alternative derivation of the evolution equation (3.22). We have benefited greatly from stimulating discussions with N. O. Weiss. E. K. is grateful for support from the Harvard Society of Fellows and St. John's College, Cambridge.

Appendix. Derivation of the functions $I_k(m)$, $J_k(m)$, $k = 1,2,3$.

(*a*) *Thermohaline case*

From (2.44) we have

$$\frac{d\mathscr{E}}{dT^*} = \langle a' F(a) \rangle \tag{A 1}$$

or

$$N^2 \frac{d}{dT^*}\left(\frac{m}{(1+m)^2}\right) = \left(1+\frac{1+\delta}{\tau\Delta}\right)\left\langle a'a''' - a'^2 a^2\left(4+\frac{2}{\omega}\right)\right\rangle + \frac{\sigma}{\tau\Delta}\langle a'^2\rangle(1-\mu\tau^2). \tag{A 2}$$

Now at leading order $a''' = 3a^2 a' - Na'$ (from (2.35)), so the right-hand side of (A 2) can be written entirely in terms of the two averages $\langle a'^2\rangle$, $\langle a^2 a'^2\rangle$. It is clear from the solution (2.40) that the former of these is equal to N^2, and the latter equal to N^3, times a function of m alone. After some manipulation, and use of formulae given by Davis (1962), we obtain

$$\left.\begin{aligned}\langle a^2a'^2\rangle &= N^3I_1(m); \quad I_1(m) = 4\left[(m-1)(2-m)+2(1-m+m^2)\frac{E(m)}{K(m)}\right]\Big/15(1+m)^3,\\ \langle a'^2\rangle &= N^2J_1(m); \quad J_1(m) = 2\left[(m-1)+(m+1)\frac{E(m)}{K(m)}\right]\Big/3(1+m)^2,\end{aligned}\right\} \tag{A 3}$$

where $K(m)$, $E(m)$ are the complete elliptic integrals of the first and second kinds respectively. Substitution into (A 2) then yields (2.45).

(b) *Magnetic case*

From (3.28) we can see that as in the previous case the important averaged quantities are once again $\langle a^2a'^2\rangle$ and $\langle a'^2\rangle$. The type of solution depends on the parameters M and N.

(i) For snoidal solutions, M, $N > 0$; then

$$\langle a^2a'^2\rangle = MN^3I_1(m), \quad \langle a'^2\rangle = MN^2J_1(m), \tag{A 4}$$

where $I_1(m)$, $J_1(m)$ are defined as above.

(ii) For cnoidal solutions, $M < 0$ and N can take either sign; then

$$\left.\begin{aligned}\langle a^2a'^2\rangle &= |MN^3|\,I_2(m);\\ &I_2(m) = 4\left[(m-1)(2-m)+2(1-m+m^2)\frac{E(m)}{K(m)}\right]\Big/15|1-2m|^3,\\ \langle a'^2\rangle &= |MN^2|\,J_2(m);\\ &J_2(m) = 2\left[(1-m)+(2m-1)\frac{E(m)}{K(m)}\right]\Big/3(1-2m)^2.\end{aligned}\right\} \tag{A 5}$$

(iii) For dnoidal solutions, $M < 0, N > 0$; then

$$\left.\begin{aligned}\langle a^2a'^2\rangle &= -MN^3I_3(m);\\ &I_3(m) = 4\left[(m-1)(2-m)+2(1-m+m^2)\frac{E(m)}{K(m)}\right]\Big/15(2-m)^3,\\ \langle a'^2\rangle &= -MN^2J_3(m);\\ &J_3(m) = 2\left[2(m-1)+(2-m)\frac{E(m)}{K(m)}\right]\Big/3(2-m)^2.\end{aligned}\right\} \tag{A 6}$$

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